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The Landau–Lifshitz equation describes the Ising spin correlation function in the free-fermion model

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Abstract. We consider time and space dependence of the Ising spin correlation function in a continuous one-dimensional free-fermion model. By the Ising spin we imply the 'sign' variable, which takes alternating ± 1 values in adjacent domains bounded by domain walls (fermionic world paths). The two-point correlation function is expressed in terms of the solution of the Cauchy problem for a nonlinear partial differential equation, which is proved to be equivalent to the exactly solvable Landau–Lifshitz equation. A new zero-curvature representation for this equation is presented.

In turn, the initial condition for the Cauchy problem is given by the solution of a nonlinear ordinary differential equation, which has also been derived. In the Ising limit the above-mentioned partial and ordinary differential equations reduce to the sine-Gordon and Painlevé III equations, respectively.

1. Introduction

In this paper we continue to study correlation properties of spinless nonrelativistic fermions, which propagate in a line and can appear and annihilate in pairs. The model Hamiltonian is given by [1]

$$\mathcal{E} = \int_{-\infty}^{\infty} \mathrm{d}x \left\{ \mu \psi^+ \psi + \frac{\mathrm{d}\psi^+}{\mathrm{d}x}\frac{\mathrm{d}\psi}{\mathrm{d}x} + \mathrm{i}g \left(\frac{\mathrm{d}\psi^+}{\mathrm{d}x}\psi^+ + \frac{\mathrm{d}\psi}{\mathrm{d}x}\psi \right) \right\} \tag{1.1}$$

where $\psi(x)$ and $\psi^+(x)$ are canonical Fermi fields: $\{\psi(x), \psi^+(x')\} = \delta(x - x')$. The chemical potential μ takes discrete ± 1 values, whereas the parameter g is continuous: $0 \leqslant g \leqslant \infty$. A typical pattern of fermionic world paths is shown in figure 1. Well known parallelism of the quantum field theory and statistical mechanics [2, 3] allows one to interpret such patterns as possible configurations of domain walls in a two-dimensional statistical mechanical system. The Euclidean time variable τ is then treated as the second space variable in the plane. Let us suppose $\mathbb{Z}_2$ symmetry is broken in domains, and denote by $\sigma(x, \tau)$ the corresponding order parameter—Ising spin, which takes alternating ± 1 values in adjacent domains (see figure 1). The subject of our interest is the two-point correlation function of the Ising spin order parameter

$$\mathcal{P}(x, \tau) = \langle \sigma(x, \tau)\sigma(0, 0)\rangle. \tag{1.2}$$

Here $\langle\rangle$ denotes either the ground-state average in the fermionic model, or the Gibbs average in its statistical mechanical counterpart.

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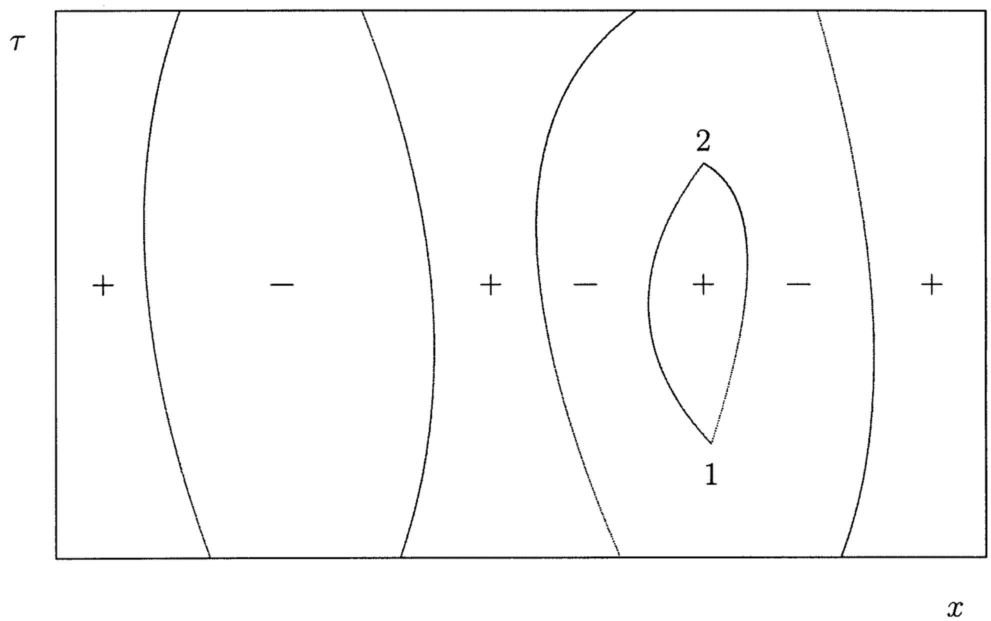

Figure 1. Fermion world paths (= domain walls) and Ising spins corresponding to Hamiltonian (1.1). At points 1 and 2 the total number of fermions changes.

The problem outlined above is motivated by the theory of the incommensurate soliton liquid phase, which takes place in lattices of atoms adsorbed on a crystalline substrate [4, 5]. In the simplest case the substrate forms an anisotropic periodical potential relief for adsorbed atoms, say $V_0 \cos(2\pi x/b)$. The commensurate phase is characterized by an integer p equal to the ratio of the lattice periods of the adsorbed atoms and the substrate potential. If $p = 2$, there are two equivalent-in-energy configurations of the overlay lattice shifted with respect to one another by the substrate lattice period b. So, the commensurate phase being $\mathbb{Z}_2$ degenerated can be characterized by the Ising spin order parameter $\sigma \equiv \exp(\pi \mathrm{i} u_x/b) = \pm 1$, where u_x is the displacement of the lattice of adsorbed atoms.

As the concentration of adsorbed atoms changes, they undergo the transition into the incommensurate phase. Closely enough to the transition point, the incommensurate phase may be conceived as commensurate regions where $\sigma = 1$ and $\sigma = -1$ separated by domain walls. At zero temperature, domain walls are parallel to the y-axis and form a periodical lattice in the x-direction incommensurate with the substrate periodicity. At a finite temperature, domain walls bend and collide with each other. So, the generic state looks like the pattern in figure 1. Points 1, 2 in figure 1, where a domain wall turns backward, correspond to dislocations in the lattice of adsorbed atoms [5, 6]. It is clear, that fermionic models provide a convenient and powerful tool for looking into the physics discussed above, and indeed they have been widely used [5–8]. Calculations of thermodynamic quantities and correlation functions of fermionic fields (i.e. fermionic Green functions) become especially simple in free-fermion models. On the other hand, the observable quantity is the Ising spin order parameter $\sigma(x, \tau)$, rather than fermionic fields. Indeed, x-ray or neutron scattering experiment measures the $\boldsymbol{k}$-dependent susceptibility $\chi(\boldsymbol{k})$ being just the Fourier transfer of the correlation function (1.2).

Generally speaking, interpretation of scattering experiments in incommensurate crystals is known to be quite a difficult problem, especially near the phase transition point [9, 10]. So, it is desirable to find the exact correlation function $\mathcal{P}(x, \tau)$ and its Fourier transform $\chi(\boldsymbol{k})$ for the soliton liquid phase, at least in a simple free-fermion model (1.1).

Before proceeding to the results obtained, it is worthwhile to make two notations concerning this model.

(i) Values $\mu = +1$ and $\mu = -1$ of the chemical potential correspond to ordered (commensurate), and disordered (incommensurate) phases, respectively. A well pronounced soliton lattice appears in the disordered phase $\mu = -1$ for small enough values of g: $g \ll 1$. Correlation function (1.2) in this case is essentially anisotropic and oscillates in x. For $g = 0$ it is known [11], that the leading term in the large-distance asymptotics of (1.2) is proportional to $x^{-1/2} \sin x$. In the opposite limit $g \to \infty$, fermionic model (1.1) is equivalent to the two-dimensional Ising model in the critical region. Here correlation function (1.2) reduces to the well known expressions obtained by McCoy *et al* [12].

(ii) The free-fermion model we consider is closely related with several models studied in the literature. We shall mention two of them. First, model (1.1) is equivalent to the XY spin chain in the double scaling limit introduced by Vaidya and Tracy [13]. These authors discovered oscillatory behaviour of the correlation functions in the double scaling XY model. Jimbo *et al* [14] calculated n-point correlation functions of Pauli matrices in this model and established their relationship to some ordinary differential equation. In the limit $g \to 0$ their equation reduces to the Painlevé V transcendent. For further details see [1]. Second, model (1.1) is just the continuous limit of the discrete fermionic model used by Bohr [6] to describe the effect of dislocations on the commensurate–incommensurate phase transition near the point $p = 2$. In particular, Bohr calculated the average Ising spin value $\langle\sigma\rangle$ in the ordered (commensurate) phase $\mu = 1$. In the continuous limit his result reduces to $\langle\sigma\rangle = (1 + g^2)^{-1/8}$.

Several techniques have been developed to determine correlation functions in exactly solvable models [15, 16]. One of them [16] involves three steps. (i) The correlation function is expressed as the determinant of a Fredholm integral operator. (ii) Determinant representation is then used to write down a nonlinear differential (or integro-differential) equation associated with the correlation function. (iii) The large-distance asymptotics of the correlation function are analysed by use of the obtained equation and related Riemann–Hilbert problem.

In the previous paper [1] we completed step (i), and started to perform step (ii). Namely, for model (1.1) the determinant representation for the correlation function (1.2) was obtained in both ordered $\mu = 1$ and disordered $\mu = -1$ phases. In the ordered phase $\mu = 1$ we asserted without proof a relationship of the correlation function (1.2) with a certain nonlinear partial differential equation.

This paper is devoted to the second step, which is completed here for both phases $\mu = \pm 1$. The emphasis is on the more complicated and physically interesting disordered phase $\mu = -1$. In the half-plane $\tau \geqslant 0$, $-\infty < x < \infty$, we derive the Cauchy problem, which determines the correlation function $\mathcal{P}(x, \tau)$. The Cauchy problem consists of the nonlinear partial differential equation, and initial conditions for it.

The partial differential equation describing evolution of the correlation function $\mathcal{P}(x, \tau)$ in τ is the same as that obtained previously in the ordered phase [1]. We prove, that this equation is equivalent to the well known exactly solvable Landau–Lifshitz equation, which describes a classical anisotropic one-dimensional ferromagnet [17–19].

The initial condition on the line $\tau = 0$, $-\infty < x < \infty$ for the Cauchy problem is given by the solution of a nonlinear ordinary differential equation of the fourth order. In the Ising limit $g \to \infty$ it reduces to Painlevé III in agreement with McCoy *et al* [12].

The paper is organized as follows. In section 2 we describe briefly the determinant representation for $\mathcal{P}(x, \tau)$. We then express $\mathcal{P}(x, \tau)$ in terms of the solution of the related Fredholm integral equations. In section 3 we formulate the Riemann–Hilbert problem, which is equivalent to the above mentioned equations. In section 4 the obtained Riemann–Hilbert problem is used to derive a matrix identity, which provides the zero-curvature representation for an exactly solvable nonlinear partial differential equation. Similar analysis performed in section 5 for the case $\tau = 0$ leads to a nonlinear ordinary differential equation. Results obtained in sections 2–5 relate to the disordered phase $\mu = -1$. Section 6 describes their modification in the ordered phase $\mu = 1$.

We put some cumbersome formulae and extended computations into the appendices. In appendix A we prove the equivalence of the obtained nonlinear partial differential equation with the Landau–Lifshitz equation. In appendix B we derive explicit expressions for elliptic matrices providing the zero-curvature representation for the above-mentioned equation. Elliptic matrices, which are related to the ordinary differential equation described in section 5, are given in appendix C.

2. Determinant formulae and integral equations

It is usual practice to start studying correlation functions in exactly solvable models by expressing these functions as determinants of Fredholm linear integral operators [16]. For the correlation function (2.1) such a determinant representation was obtained in [1]. Here we describe this representation and then use it to express the logarithmic derivative of the correlation function in terms of solutions of linear Fredholm integral equations of the second kind.

First of all, a remark concerning notation is necessary. The Hamiltonian (1) in [1] seems at first sight somewhat more general than (1.1). However, the former reduces to the latter after rescaling $x \to x l_0$, $\psi \to \psi/\sqrt{l_0}$, $\psi^+ \to \psi^+/\sqrt{l_0}$, $\mathcal{E} \to \mathcal{E}|\Omega|$, if $l_0 = (s/|\Omega|)^{1/2}$, $\mu = \mathrm{sign}(\Omega)$, $g = \Gamma/(4s|\Omega|)^{1/2}$.

We denote by $\hat{\sigma}(x)$ the unitary operator $\hat{\sigma}(x)$, corresponding to the Ising spin order parameter $\sigma(x, \tau)$. This operator can be written in terms of the fermionic fields [1]

$$\hat{\sigma}(x) = \exp\left\{\mathrm{i}\pi \int_{-\infty}^{x} \mathrm{d}x' \psi^+(x')\psi(x')\right\}$$

where the boundary condition $\sigma(x, \tau) \to 1$ for $x \to -\infty$ is supposed. Relation (1.2) then takes the form

$$\mathcal{P}(x, \tau) = \langle\Phi|\hat{U}(-\tau)\hat{\sigma}(x)\hat{U}(\tau)\hat{\sigma}(0)|\Phi\rangle. \tag{2.1}$$

Here $|\Phi\rangle$ denotes the Hamiltonian ground state, $\hat{U}(\tau) = \mathrm{e}^{-\tau\mathcal{E}}$ is the Euclidean evolution operator.

The determinant representation for the correlation function (2.1) in the disordered phase $\mu = -1$ reads as [1]: $\mathcal{P}(x, \tau) = \lim_{\delta\to 0} \mathcal{P}_\delta(x, \tau)$, where

$$\mathcal{P}_\delta(x, \tau) = C(\delta) \det[1 + \hat{D}_\delta(x, \tau)].$$

Here $C(\delta)$ is the constant factor, $\hat{D}_\delta(x, \tau)$ is the linear integral operator acting on a function $f(p)$ in the following way

$$\hat{D}_\delta(x, \tau) f(p) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} D_{pk}\, f(k) \exp\{-\tfrac{1}{2}\mathrm{i}x(p + k) - \tfrac{1}{2}\tau[w(p) + w(k)]\}$$

with

$$D_{pk} = \frac{1}{p+k}\left[\frac{A(k)}{A(p)} - \frac{A(p)}{A(k)}\right] \qquad A(p) = \left[\frac{w(p)}{p^2+\delta^2}\right]^{1/2}$$
$$w(p) = [(\mu + p^2)^2 + (2gp)^2]^{1/2}.$$

The well known procedure [14] allows one to obtain Fredholm integral equations from determinant formulae. Denote by Tr the trace of a linear integral operator

$$\mathrm{Tr}\,\hat{L} = \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} L_{pk}|_{k=p}$$

where L_{pk} is the operator kernel:

$$\hat{L} f(p) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} L_{pk} f(k).$$

Differentiating the equality $\ln\det[1 + \hat{D}_\delta(x,\tau)] = \mathrm{Tr}\ln[1 + \hat{D}_\delta(x,\tau)]$, we obtain

$$\frac{\partial}{\partial x}\ln \mathcal{P}_\delta(x,\tau) = \mathrm{Tr}\left((1 + \hat{D}_\delta(x,\tau))^{-1}\frac{\partial \hat{D}_\delta(x,\tau)}{\partial x}\right). \tag{2.2}$$

It is essential for further analysis, that the kernel of the operator $\partial\hat{D}_\delta(x,\tau)/\partial x$ factorizes

$$\left(\frac{\partial \hat{D}_\delta(x,\tau)}{\partial x}\right)_{pk} = \frac{\mathrm{i}}{2}[a_1(p)a_2(k) - a_2(p)a_1(k)] \tag{2.3}$$

where

$$a_1(p) = A(p)\exp\left[-\frac{\mathrm{i}xp}{2} - \frac{\tau w(p)}{2}\right] \qquad a_2(p) = [A(p)]^{-1}\exp\left[-\frac{\mathrm{i}xp}{2} - \frac{\tau w(p)}{2}\right].$$

Combining (2.2) and (2.3) we obtain

$$\frac{\partial}{\partial x}\ln \mathcal{P}_\delta(x,\tau) = \frac{\mathrm{i}}{2}\int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi}[u_1(p)a_2(p) - u_2(p)a_1(p)] \tag{2.4}$$

where $u_j(p)$, $j = 1, 2$ denote solutions of Fredholm integral equations of the second kind:

$$(1 + \hat{D}_\delta(x,\tau))u_j = a_j \qquad j = 1, 2. \tag{2.5}$$

3. Riemann–Hilbert problems

In this section we derive a Riemann–Hilbert problem on the complex torus, which is equivalent to integral equations (2.5), and then proceed to the limit $\delta \to 0$.

First, let us rewrite equations (2.5) in terms of new variables. Set

$$v_j(p) = u_j(p)A(p)\exp\left[\frac{\mathrm{i}xp}{2} + \frac{\tau w(p)}{2}\right]$$
$$z(p) = \int_0^p \frac{\mathrm{d}k}{w(k)} \qquad 2\omega = \int_{-\infty}^{\infty}\frac{\mathrm{d}p}{w(p)} \qquad \omega' = \int_{-p_1}^{p_1}\frac{\mathrm{d}p}{w(p)}$$
$$p_1 = \begin{cases} \mathrm{i}(g - \sqrt{g^2-1}) & \text{for } g \geqslant 1 \\ \mathrm{i}g + \sqrt{1-g^2} & \text{for } 0 < g < 1. \end{cases} \tag{3.1}$$

Here z is the new independent variable, 2ω and $2\omega'$ are the primitive periods of the elliptic function $p(z)$, p_1 is one of four zeros of the function $w^2(p) = (p^2-1)^2 + (2gp)^2$. For

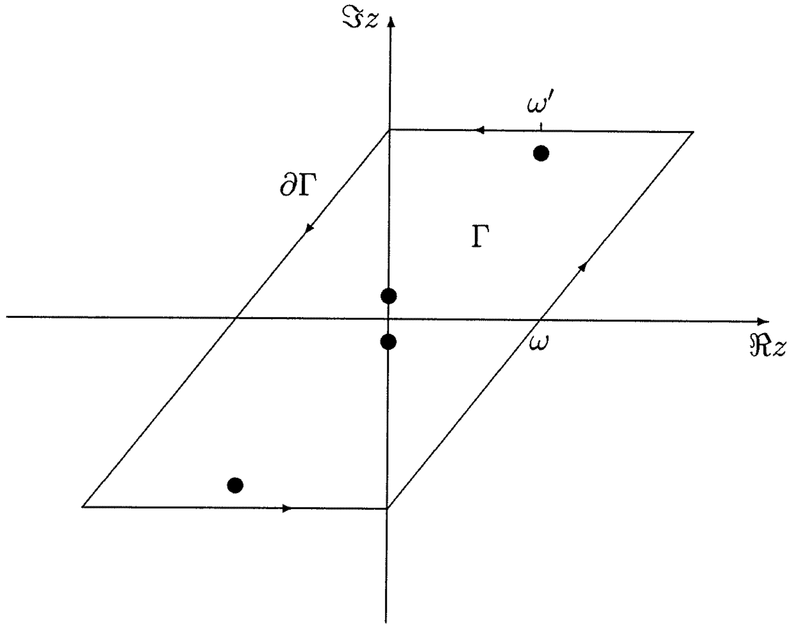

Figure 2. Fundamental region Γ for quasiperiodical functions $v_j(z)$. Poles of $v_j(z)$ are indicated.

$g \geqslant 1$ the period $2\omega'$ is purely imaginary, $\Im\omega' > 0$, $\Re\omega' = 0$. For $0 < g < 1$ the period $2\omega'$ gains the real part: $\Re\omega' = \omega$. In the new notation equations (2.5) read as

$$v_j(z) = \tilde{a}_j(z) + \int_{-\omega}^{\omega} \frac{dz'}{2\pi} \frac{w(z')}{p(z') + p(z)} \left[\frac{A^2(z)}{A^2(z')} - 1 \right] \exp[-ixp(z') - \tau w(z')] v_j(z') \qquad (3.2)$$

where $j = 1, 2$, $\tilde{a}_1(z) = A^2(z)$, $\tilde{a}_2(z) = 1$.

Further, denote by Γ the open region in the z-plane bounded by the parallelogram of periods shown in figure 2.

Lemma. Let $\tau > 0$. Then functions $v_j(z)$, $j = 1, 2$ which solve equations (3.2) can be analytically continued from the interval $(-\omega, \omega)$ into the whole z-plane, where they obey the following conditions.

(i) $v_j(z + 2\omega) = v_j(z)$.

(ii) $v_j(z - 2\omega') = v_j(z) - 2i v_j(z - \omega') \exp[ixp(z) + \tau w(z)]$.

(iii) Functions $v_j(z)$ are meromorphic in the open region Γ.

(iv) Functions $v_j(z)$ have exactly four simple poles in Γ in the points $b_{1,2} = \pm i\alpha$, $b_{3,4} = \pm(\omega' - i\alpha)$, which are the zeros of the function $[p^2(z) + \delta^2]$: $p(b_1) = i\delta$. Moreover,

$$\mathrm{Res}_{z=b_1} v_j(z) = \mathrm{Res}_{z=b_3} v_j(z) \qquad \mathrm{Res}_{z=b_2} v_j(z) = \mathrm{Res}_{z=b_4} v_j(z).$$

(v) Except for the poles b_i, $i = 1, \ldots, 4$, functions $v_j(z)$ are continuous in $\overline{\Gamma}$.

(vi) $v_j(\omega) = 1$, $v_j(\omega - \omega' + i0) = (-1)^j$.

Conversely, a pair of functions obeying (i)–(vi) solve equations (3.2).

Remarks.

(1) By virtue of (ii), functions $v_j(z)$ have essential singularities at the points $(2n+1)\omega + m\omega'$, $n, m \in \mathbb{Z}$, $m \neq 0$.

(2) For $z \to \omega + \mathrm{i}0$ statement (vi) can be further detailed:

$$
\begin{aligned}
v_j(z) &= 1 - (z-\omega)\int_{-\infty}^{\infty}\frac{\mathrm{d}p}{2\pi}u_j(p)[a_2(p) - a_1(p)] + \mathrm{O}(z-\omega)^2 \\
v_j(z-\omega') &= (-1)^j - (z-\omega)\int_{-\infty}^{\infty}\frac{\mathrm{d}p}{2\pi}u_j(p)[a_2(p) + a_1(p)] + \mathrm{O}(z-\omega)^2.
\end{aligned}
\tag{3.3}
$$

(3) For real x and τ, functions $v_j(z)$ obey the symmetry property $v_j(z; x, \tau) = v_j^*(z^*, -x, \tau)$, where * denotes the complex conjugation.

Proof of the direct statement of the lemma is straightforward. To prove the converse statement, we rewrite the integral on the right-hand side of (3.2) as

$$
\begin{aligned}
&\int_{-\omega+\epsilon}^{\omega+\epsilon}\frac{\mathrm{d}z'}{4\pi\mathrm{i}}\frac{w(z')}{p(z')+p(z)}\left[\frac{A^2(z)}{A^2(z')} - 1\right][v_j(z'+\omega') - v_j(z'-\omega')] \\
&\qquad = \int_{\partial(\Gamma+\epsilon)}\frac{\mathrm{d}z'}{4\pi\mathrm{i}}\frac{w(z')}{p(z)-p(z')}\left[-\frac{A^2(z)}{A^2(z')} - 1\right]v_j(z').
\end{aligned}
\tag{3.4}
$$

Here ϵ is a small positive number, the parallelogram $(\Gamma + \epsilon)$ is obtained from Γ by translation in z' by ϵ, $\partial(\Gamma + \epsilon)$ denotes the parallelogram boundary. In deriving (3.4) we used (i), (ii) and the translation property common for the functions $p(z), w(z)$, and $A^2(z)$: $f(z + \omega') = -f(z)$, where $f = p, w, A^2$. Nine poles (in the z' variable) of the integrand on the right-hand side of (3.4) are b_i, $i = 1, \ldots, 9$. Four of them were given in (iv), and the remaining five are

$$
b_5 = z \qquad b_6 = \omega' - z \qquad b_7 = \omega \qquad b_8 = \omega - \omega' \qquad b_9 = \omega + \omega'.
$$

Among these poles only b_5, b_7, b_8 and b_9 contribute to the integral. Really, residues in $b_{1,2}$ cancel with those in $b_{3,4}$ owing to (iv), and the residue in b_6 vanishes since $A^2(z) = -A^2(\omega' - z)$. Contributions from poles b_5 and b_7 give $v_j(z)$ and $[-1 - A^2(z)]/2$, respectively. We have taken into account (vi) and equality $A^2(\omega) = 1$. A more subtle analysis is needed for poles in b_8 and b_9, which lie in the integration contour $\partial(\Gamma + \epsilon)$. Near these points, integrals in (3.4) must be understood in the sense of the Cauchy principal value. Alternatively, we may divide poles in b_i, $i = 8, 9$ into two parts according to the Plemel formula

$$
\frac{1}{z - b_i} \to \frac{1}{2}\left[\frac{1}{z - b_i + \mathrm{i}0} + \frac{1}{z - b_i - \mathrm{i}0}\right].
$$

Poles in $(b_8 - \mathrm{i}0)$ and in $(b_9 + \mathrm{i}0)$ are removed from $(\Gamma + \epsilon)$. The total contribution of poles in $(b_8 + \mathrm{i}0)$ and in $(b_9 - \mathrm{i}0)$ is $[-1 + A^2(z)](-1)^j/2$. Collecting all terms coming from b_5, b_7, $(b_8 + \mathrm{i}0)$ and $(b_9 - \mathrm{i}0)$, we obtain $v_j(z) - \tilde{a}_j(z)$, completing the proof.

Thus, conditions (i)–(vi) characterize functions $v_j(z)$ completely. These conditions can be reformulated as a Riemann–Hilbert problem. We set

$$
X(z) = \frac{1}{2}\begin{pmatrix} v_1(z) & v_2(z) \\ v_1(z-\omega') & v_2(z-\omega') \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.
$$

Denote by Γ_+ the upper half of the parallelogram Γ, i.e. $\Gamma_+ = \{z \in \Gamma | \Im z > 0\}$. In terms of the matrix $X(z)$ statements (i)–(vi) of the lemma transforms to the following.

Problem I.

(i) $X(z + 2\omega) = X(z)$.

(ii) $X(z-\omega') = M(z)X(z)$ where

$$M(z) = \begin{pmatrix} 0 & 1 \\ 1 & -2\mathrm{i}\exp[\mathrm{i}xp(z) + \tau w(z)] \end{pmatrix}. \tag{3.5}$$

(iii) $X(z)$ is meromorphic in the open region Γ_+.

(iv) $X(z)$ has exactly two simple poles in Γ_+ at the points $b_1 = \mathrm{i}\alpha$ and $b_3 = \omega' - \mathrm{i}\alpha$, which are the zeros of the function $[p(z) - \mathrm{i}\delta]$. Moreover, $\mathrm{Res}_{z=b_1} X(z) = \mathrm{Res}_{z=b_3} X(z)$.

(v) Except for the poles b_1, b_3, $X(z)$ is continuous in $\overline{\Gamma}_+$.

(vi) $X(\omega + \mathrm{i}0) = I$, where I is the unit matrix.

Conditions (i)–(vi) for the matrix $X(z)$ define the Riemann–Hilbert problem in the complex torus. By construction, this problem is equivalent to the integral equations (3.2). Let

$$X(z) = I + \sum_{m=1}^{\infty} \frac{(z-\omega)^m}{m!} X_m \tag{3.6}$$

be the asymptotic expansion of the matrix $X(z)$ at $z = \omega$ for $z \in \Gamma_+$. Then, combining (2.4) and (3.3), we have

$$\frac{\partial}{\partial x} \ln \mathcal{P}_\delta(x,\tau) = -\frac{\mathrm{i}}{2} \mathrm{tr}(X_1\sigma_3) \tag{3.7}$$

where $\mathrm{tr}\, A$ denotes the trace of a 2×2 matrix A: $\mathrm{tr}\, A = A_{11} + A_{22}$. We use the standard notation for the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now let us proceed to the limit $\delta \to 0$. It is clear, that in this limit the matrix $X(z; x, \tau)$ still satisfies conditions (i), (ii) and (vi). However, conditions (iii) and (iv) must be modified, since the poles located at $\pm\mathrm{i}\alpha$ merge to the origin to the limit $\delta \to 0$. One could expect to obtain the second-order pole as a result. However, accurate analysis of equations (3.2) shows that in fact the limiting matrix $X(z; x, \tau)$ has the first-order pole at the origin $z = 0$.

Thus, the Riemann–Hilbert problem which defines the matrix $X(z)$ in the limit $\delta \to 0$ is as follows.

Problem II.

(i) $X(z + 2\omega) = X(z)$.

(ii) $X(z - \omega') = M(z)X(z)$ where $M(z)$ is given by (3.5).

(iii) $X(z)$ is meromorphic in the open region Γ_1 shown in figure 3.

(iv) The single pole of $X(z)$ in Γ_1 is simple and lies in the origin $z = 0$.

(v) Except for the origin $z = 0$, $X(z)$ is continuous in $\overline{\Gamma}_1$.

(vi) $X(\omega + \mathrm{i}0) = I$.

Remark. If $\tau = 0$ and $x > 0$, then $X(z) = \sigma_1 X^*(z^*)\sigma_1$.

In the limit $\delta \to 0$, relation (3.7) expresses the logarithm derivative of the correlation function $\mathcal{P}(x,\tau)$ in terms of the solution of the Riemann–Hilbert problem II. From here on we shall refer (i)–(vi) to corresponding items in problem II.

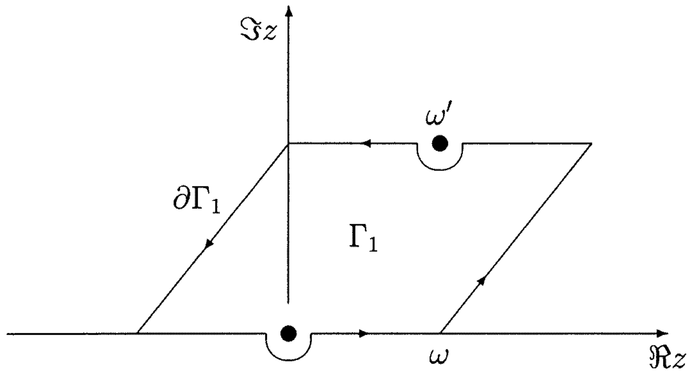

Figure 3. Fundamental region Γ_1 for matrix $X(z)$ in the limit $\delta \to 0$. Poles of $X(z)$ are indicated.

4. Connection with partial differential equations

In this section we study the dependence on x and τ of the matrix $X(z; x, \tau)$, which solves problem II. In what follows we shall also use the abbreviated notation $X(z)$ for this matrix.

First, let us consider the function $\det X(z)$. Since $\det M(z) = -1$, conditions (i)–(vi) ensure, that $\det X(z)$ is the elliptic function having in Γ_1 a single second-order pole at the origin. Translation properties of $\det X(z)$ are: $\det X(z) = \det X(z+2\omega) = -\det X(z+\omega')$. Therefore, $\det X(z)$ can be factorized as

$$\det X(z; x, \tau) = C(x, \tau)\vartheta_1\left[\frac{z - z_1(x, \tau)}{2\omega}\right]\vartheta_1\left[\frac{z - z_2(x, \tau)}{2\omega}\right][\vartheta_1(z/2\omega)]^{-2} \tag{4.1}$$

with

$$\vartheta_1(u) = \mathrm{i}\exp\left[-\mathrm{i}\pi\left(u - \frac{\tilde{\tau}}{4}\right)\right]\vartheta_3\left(u + \frac{1-\tilde{\tau}}{2}\right) \qquad \vartheta_3(u) = \sum_{n\in\mathbb{Z}}\exp(2\pi\mathrm{i}nu + \pi\mathrm{i}n^2\tilde{\tau}) \tag{4.2}$$

and

$$z_1(x, \tau) + z_2(x, \tau) = \omega. \tag{4.3}$$

Here $\vartheta_{1,3}(u)$ are θ-functions with the quasiperiod $\tilde{\tau} = \omega'/(2\omega)$, the coefficient $C(x, \tau)$ is determined by the relation $\det X(\omega + \mathrm{i}0) = 1$. So, the matrix $X(z)$ degenerates (having rank 1) at the points $z_{1,2}(x, \tau) + 2n\omega + m\omega'$, $(m, n \in \mathbb{Z})$, which are related by (4.3) and move under changes of x and τ.

Define four 2×2 matrices $E(z; x, \tau)$, $Y(z; x, \tau)$, $U(z; x, \tau)$, $V(z; x, \tau)$, or in abbreviated notation, $E(z)$, $Y(z)$, $U(z)$, $V(z)$:

$$E(z) = \exp\left(-\frac{\sigma_3}{2}(\mathrm{i}xp(z) + \tau w(z))\right) \qquad Y(z) = E(z)X(z) \tag{4.4}$$

$$U(z) = Y^{-1}(z)\frac{\partial Y(z)}{\partial x} = -\frac{\mathrm{i}p(z)}{2}X^{-1}(z)\sigma_3 X(z) + X^{-1}(z)\frac{\partial X(z)}{\partial x} \tag{4.5}$$

$$V(z) = Y^{-1}(z)\frac{\partial Y(z)}{\partial \tau} = -\frac{w(z)}{2}X^{-1}(z)\sigma_3 X(z) + X^{-1}(z)\frac{\partial X(z)}{\partial \tau}. \tag{4.6}$$

Translation properties of the above matrices are

$$E(z) = E(z+2\omega) = [E(z+\omega')]^{-1}$$

$$Y(z) = Y(z+2\omega) \qquad Y(z-\omega') = Y(z)\begin{pmatrix} 0 & 1 \\ 1 & -2\mathrm{i} \end{pmatrix}$$

$$U(z) = U(z+2\omega) = U(z+\omega') \tag{4.7}$$

$$V(z) = V(z+2\omega) = V(z+\omega'). \tag{4.8}$$

Moreover, matrices $U(z)$ and $V(z)$ are meromorphic in the z-plane and, hence, elliptic. Relations (4.5), (4.6) can be rewritten

$$\frac{\partial Y(z;x,\tau)}{\partial x} = Y(z;x,\tau)U(z;x,\tau) \qquad \frac{\partial Y(z;x,\tau)}{\partial \tau} = Y(z;x,\tau)V(z;x,\tau)$$

as a system of linear differential equations for the matrix $Y(z;x,\tau)$. Its compatibility condition

$$\frac{\partial U(z;x,\tau)}{\partial \tau} - \frac{\partial V(z;x,\tau)}{\partial x} + [V(z;x,\tau), U(z;x,\tau)] = 0 \tag{4.9}$$

has the form of the zero-curvature relation, which is crucial in the Faddeev's school version of the inverse scattering method [17].

It follows from (4.5), (4.6), (4.1) that elliptic matrices $U(z)$, $V(z)$ have in Γ_1 a stable pole in ω, and two simple moveable poles at the points $z_1(x,\tau)$, $z_2(x,\tau)$ related by (4.3). Combining (vi) with (4.4)–(4.8), we come to representations

$$U(z) = \frac{\mathrm{i}}{2}\sigma_3\zeta(z-\omega) - P_1\frac{\partial z_1}{\partial x}\zeta(z-z_1) - P_2\frac{\partial z_2}{\partial x}\zeta(z-z_2) + U_0 \tag{4.10}$$

$$V(z) = -\frac{1}{2}\sigma_3\wp(z-\omega) + B\zeta(z-\omega) - L_1\frac{\partial z_1}{\partial \tau}\zeta(z-z_1) - L_2\frac{\partial z_2}{\partial \tau}\zeta(z-z_2) + V_0 \tag{4.11}$$

with

$$P_1\frac{\partial z_1}{\partial x} + P_2\frac{\partial z_2}{\partial x} = \frac{\mathrm{i}\sigma_3}{2} \tag{4.12}$$

$$L_1\frac{\partial z_1}{\partial \tau} + L_2\frac{\partial z_2}{\partial \tau} = B. \tag{4.13}$$

Here $\wp(z)$ is the Weierstrass elliptic function with primitive periods 2ω and ω', and $\zeta(x)$ is the corresponding ζ-function. Since rank $Y(z_j) = 1$, $j = 1, 2$, operators P_j, L_j are projectors: $P_j^2 = P_j$, $L_j^2 = L_j$. Furthermore, images of P_j and L_j coincide:

$$\operatorname{Im} P_j = \operatorname{Im} L_j = \ker Y(z_j) = \ker X(z_j). \tag{4.14}$$

Relations (4.3), (4.5), (4.6), (4.9), (4.1) allow us to obtain explicit expressions (B.8)–(B.12) for matrices P_j, L_j $(j = 1, 2)$, U_0, V_0. As shown in appendix B, these matrices depend on x and τ via two functions $z_1(x,\tau)$, and $\alpha(x,\tau)$, which solve the following dynamic equations

$$\dot{z}_1 = -4r^2\alpha' z_1' \tag{4.15}$$

$$\dot{\alpha} = \frac{1}{2}[\wp(z_1) + \wp(z_1-\omega) + \wp(\omega)] + \frac{f}{16(z_1')^2r^2}\left[\frac{z_1''}{2rz_1'} - 2r\frac{\partial \ln\lambda}{\partial x}\right]$$
$$+\frac{f^2}{2} - \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{f}{r}\right) - (1+2r^2)\left(\frac{\partial\alpha}{\partial x}\right)^2 \tag{4.16}$$

where r, f and λ are given by (B.14). Partial derivatives with respect to x and τ are denoted by prime and dot, respectively.

The inverse statement is also valid.

Theorem. Let functions $z_1(x,\tau)$ and $\alpha(x,\tau)$ solve equations (4.15), (4.16). Let 2×2 matrices $U(z;x,\tau)$, $V(z;x,\tau)$ be given by relations (4.10), (4.11), (B.8)–(B.12), (4.3). Then these matrices obey (4.9).

So, equations (4.15), (4.16) allow the zero-curvature representation (4.9), which indicates their integrability by the inverse scattering method [17]. The standard procedure described in [17] leads to the infinite set of integrals of motion: I_n, $n = 1,\ldots,\infty$. The first one is given by

$$I_1 = \int_{-\infty}^{\infty} \mathrm{d}x\, H(x) \tag{4.17}$$

where

$$\begin{aligned} H(x) &= \frac{1}{2}\left\{\frac{\pi^2[1+4(z_1')^2]}{4} + f^2 - \wp(z_1) - \wp(z_1-\omega) - \wp(\omega)\right\} \\ \pi(x,\tau) &= \mathrm{i}\alpha'(x,\tau)/z_1'(x,\tau). \end{aligned} \tag{4.18}$$

Up to this point we considered evolution in the Euclidean time τ. Now it is convenient to pass to the ‘Minkowski’ case replacing everywhere τ by $-\mathrm{i}t$, where t is the new (real) time variable. Consider (4.17) as the Hamiltonian of a classical evolution model with coordinate $z_1(x)$ and momentum $\pi(x)$ variables. The canonical Poisson brackets are supposed

$$\{\pi(x_1), z_1(x_2)\} = \delta(x_1 - x_2).$$

As one can verify by straightforward calculation, Hamiltonian equations of the model (4.17), (4.18) are equivalent to equations (4.15), (4.16).

The Hamiltonian density (4.18) can be rewritten in a more symmetric form

$$H(x) = \frac{\pi^2[1+4(z_1')^2]}{8} + \frac{2(z_1'')^2}{1+4(z_1')^2} + \frac{1+12(z_1')^2}{8}\tilde{\wp}\left(z_1 - \frac{\omega}{2}\right) - \frac{\tilde{\wp}(\omega/2)}{8} + \frac{\partial^2 \ln\lambda}{\partial x^2} \tag{4.19}$$

where $\tilde{\wp}(z)$ denotes the Weierstrass function with periods $\omega, \omega'/2$. The two last terms in (4.19) do not contribute to the equations of motion.

Another convenient parametrization of the model (4.17), (4.18) is

$$\begin{aligned} u &= \frac{1}{2g}[\wp(z_1) + \wp(z_1-\omega) + \wp(\omega)]^{1/2} = \frac{1}{2g}\frac{\mathrm{d}\ln p(z_1)}{\mathrm{d}z_1} \qquad k^2 = \frac{g^2}{g^2+\mu} \\ x &= \frac{\tilde{x}}{2\sqrt{g^2+\mu}} \qquad t = \frac{\tilde{t}}{4(g^2+\mu)} \\ L &= \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}\tilde{x}\left\{\frac{\dot{u}^2 - [u'' + \frac{1}{8}\mathrm{d}R/\mathrm{d}u]^2}{[(u')^2 + \frac{1}{4}R]} + k^2u^2\right\} \end{aligned} \tag{4.20}$$

where

$$R(u) = (1-u^2)(1-k^2u^2). \tag{4.21}$$

It is shown in appendix A, that the Lagrangian model (4.20) is equivalent to the complex generalization of the Landau–Lifshitz model, for which exact integrability is well known (see [17–19]). So, equations (4.15), (4.16) are equivalent to the Landau–Lifshitz equations describing a classical one-dimensional anisotropic ferromagnet. Zero-curvature representation for the Landau–Lifshitz model was obtained in papers [20, 21]. According to the above analysis, relation (4.9) with matrices (4.10), (4.11) gives an alternative zero-curvature representation for the same model. It should be noted, that our representation is quite different from that known in the literature [17]. In particular, in our representation poles of matrices $U(z)$ and $V(z)$ move in the z-plane with varying x and t. In contrast, the standard zero-curvature representation for the Landau–Lifshitz model employs matrices with immovable poles.

Returning to the correlation function $\mathcal{P}(x,\tau)$, let us differentiate relation (3.7) with respect to x, and evaluate $\operatorname{tr}\sigma_3\partial X_1/\partial x$ by the use of (B.6). After straightforward calculations we obtain, finally

$$\frac{\partial^2 \ln\mathcal{P}(x,\tau)}{\partial x^2} = -\tfrac{1}{2}H(x) \tag{4.22}$$

where $H(x)$ is given by (4.18) or (4.19). Thus, the second logarithm derivative of the quantum correlation function $\mathcal{P}(x,\tau)$ is proportional to the Hamiltonian density of the nonlinear classical model (4.17), (4.19). This model is equivalent to the Landau–Lifshitz model (A.1)–(A.3) describing the anisotropic one-dimensional ferromagnet.

5. Initial conditions

Equations (4.15), (4.16) describe evolution in τ of parameters $z_1(x,\tau)$ and $\alpha(x,\tau)$ in the region $-\infty < x < \infty$, $\tau > 0$. Here we consider x-dependence of these functions at $\tau = 0$, i.e. the initial conditions for equations (4.15), (4.16). Throughout this section parameter τ will be omitted implying $\tau = 0$.

We start from the problem II, setting $\tau = 0$, $x > 0$ in it. Let us introduce a new elliptic matrix $W(z;x)$:

$$W = Y^{-1}\frac{\partial Y}{\partial z} = -\frac{\mathrm{i}xw}{2}X^{-1}\sigma_3 X + X^{-1}\frac{\partial X}{\partial z} \tag{5.1}$$

with periods 2ω, ω'. The system of linear differential equations

$$\frac{\partial Y}{\partial x} = YU \qquad \frac{\partial Y}{\partial z} = YW$$

is compatible under the condition

$$\frac{\partial W}{\partial x} - \frac{\partial U}{\partial z} + [U,W] = 0. \tag{5.2}$$

The following expansion

$$W = -\frac{\mathrm{i}x}{2}\wp(z-\omega)\sigma_3 + A\zeta(z-\omega) - \zeta(z) + Q_1\zeta(z-z_1) + Q_2\zeta(z-z_2) + W_0 \tag{5.3}$$

with

$$A = I - Q_1 - Q_2$$

is analogous to (4.10), (4.11). Matrices Q_1, Q_2, A, W_0 depend on x, but not on z. Evaluations analogous to those described in appendix B allow one to determine the matrix $W(z,x)$, and to obtain independently the matrix $U(z;x)$. Explicit expressions for these matrices are given in appendix C. They depend on x via the function $z_1(x)$, which solves the following nonlinear ordinary differential equation of the fourth order:

$$\frac{\delta\mathcal{S}}{\delta z_1(x)} = 0 \tag{5.4}$$

where the action functional $\mathcal{S}$ is given by (cf (4.19)):

$$\mathcal{S} = \int \mathrm{d}x\, x\left\{\frac{2(z_1'')^2}{1+4(z_1')^2} + \frac{1+12(z_1')^2}{8}\tilde{\wp}\left(z_1 - \frac{\omega}{2}\right)\right\}. \tag{5.5}$$

Recall, that periods of the Weierstrass function $\tilde{\wp}(z)$ are reduced by a factor of 2 compared with those of $\wp(z)$.

It should be noted, that in the Ising limit $g \to \infty$ action (5.5) reduces to that of the Painlevé III transcendent:

$$\mathcal{S} \to \int \mathrm{d}x\, x \left[\frac{(\phi')^2}{2} + \cosh(2\phi) \right] \tag{5.6}$$

in agreement with McCoy *et al* [12]. Really, in this limit we have $\omega'/\omega \to 0$, and

$$\tilde{\wp}\left(z_1 - \frac{\omega}{2}\right) \approx \frac{8}{3}(2g^2 - 1) + 8\cosh(8gz_1) \tag{5.7}$$

for z_1 lying near the imaginary axis. Substituting $z_1(x) = \phi(x)/4g$ and (5.7) into (5.5), we obtain (5.6) to the leading (zero) order in g^{-1}†.

The point $x = 0$ is critical for the differential equation (5.4). The boundary condition at this point for the function $z_1(x)$ reads as (see appendix C):

$$z_1(x)_{x\to+0} = \frac{\omega}{2} + \sqrt{\frac{\omega x}{\pi}} \left[1 + \frac{\pi}{4\omega} \left(\frac{2\omega\eta}{\pi^2} - \frac{1}{2} \right) x + \mathrm{O}(x^2) \right] \tag{5.8}$$

where $\eta = \zeta(z; 2\omega, \omega')_{z=\omega}$.

Equation (5.4) supplied with the boundary condition (5.8) determines the function $z_1(x)$ completely in the interval $0 < x < \infty$. According to remark (3) on page 6195 and the remark on page 6196, the function $z_1(x)$ can be continued to negative x by setting $z_1(x) = z_1(-x)$. So, the initial condition $z_1(x)$ for the function $z(x, \tau)$ is determined in the whole axis $-\infty < x < \infty$. Initial conditions for another function in (4.15), (4.16) must be taken as $\alpha(x, \tau)_{\tau=0} = 0$ (see appendix C).

6. Ordered phase

So far we considered the correlation function $\mathcal{P}(x, \tau)$ in the disordered phase $\mu = -1$. However, minimal changes are needed to extend the results obtained to the ordered phase $\mu = +1$. These are listed below for completeness.

- In the ordered phase, the determinant representation for $\mathcal{P}(x, \tau)$ is given by relation (20) in [1]. Unlike the disordered phase, any limiting procedure is not needed.
- The logarithm derivative of the correlation function is still related by (3.7) to the matrix $X(z)$ solving the Riemann–Hilbert problem, which is almost the same as problem II. The only difference is in the position of the simple pole of the matrix $X(z)$. Now the pole lies at the point $z = \omega + \frac{\omega'}{2}$, instead of the origin. This implies, in turn, that degenerate points $z_{1,2}$ of the matrix $X(z)$ are now related by $z_1(x, \tau) + z_2(x, \tau) = -\omega + \omega'$. This also induces small changes in matrices $U(z), V(z), W(z)$ which we shall not specify here. It should be noted, that in the ordered phase explicit expressions for zeros of $w(p)$ change. In particular, relation (3.1) now transforms to $p_1 = \mathrm{i}(\sqrt{g^2 + 1} - 1)$.
- Relations (4.22), (4.19), (5.4), (5.5), (5.8) which connect the correlation function to the Cauchy problem are still valid. The only change is in the first term in the right-hand side of (5.8). Here $\omega/2$ should be replaced by $(-\omega + \omega')/2$.

† Similarly, the model defined by the Hamiltonian density (4.19) reduces to the sine-Gordon model in the limit $g \to \infty$. For the Landau–Lifshitz model this result is well known [17].

Appendix A

In this appendix we prove equivalence of the model defined by the Lagrangian (4.20) to the complex generalization of the Landau–Lifshitz model of the biaxial one-dimensional ferromagnet.

The Landau–Lifshitz model is defined by the Hamiltonian density [17]

$$H_{\mathrm{LL}}(x) = \frac{1}{2}\left[\left(\frac{\partial \boldsymbol{S}}{\partial x}\right)^2 - J(\boldsymbol{S})\right] \tag{A.1}$$

where $\boldsymbol{S}(x) = (S_1(x), S_2(x), S_3(x))$ is a real vector on a unit sphere $\boldsymbol{S}(x) \in \mathbf{S}^2 \subset \mathbb{R}^3$:

$$\boldsymbol{S}^2(x) = \sum_{a=1}^{3} S_a^2(x) = 1 \tag{A.2}$$

$J(\boldsymbol{S})$ is the diagonal quadratic form

$$J(\boldsymbol{S}) = J_1 S_1^2 + J_2 S_2^2 + J_3 S_3^2$$
$$J_1 \leqslant J_2 \leqslant J_3 \qquad J_1 + J_2 + J_3 = 0.$$

The equation of motion is a Hamiltonian

$$\dot{\boldsymbol{S}} = \{\mathcal{H}_{LL}, \boldsymbol{S}\}$$

where $\mathcal{H}_{\mathrm{LL}}$ is the Hamiltonian: $\mathcal{H}_{\mathrm{LL}} = \int \mathrm{d}x\, H_{\mathrm{LL}}(x)$. The Poisson structure is induced by the Poisson brackets

$$\{S_a(x), S_b(y)\} = -\epsilon_{abc} S_c(x)\delta(x-y). \tag{A.3}$$

Here ϵ_{abc} is the antisymmetric tensor normalized as $\epsilon_{123} = 1$. The model described above is known as a 'universal' (in a certain sense) integrable model, which contains sine-Gordon and nonlinear Schrödinger models as the limiting cases. We shall consider the complex generalization of the model (A.1)–(A.3) supposing, that $\boldsymbol{S}(x)$ is a vector on a *complex* unit sphere (A.2): $\mathbf{S}^2 \subset \mathbb{C}^3$.

First, let us introduce variables $\Psi(x)$ and $\Psi^*(x)$ on the sphere by the relations

$$\Psi = \frac{S_1 + \mathrm{i}S_2}{(1-S_3)^{1/2}} \qquad \Psi^* = \frac{S_1 - \mathrm{i}S_2}{(1-S_3)^{1/2}}.$$

Since the Poisson bracket on these functions is canonical

$$\{\Psi(x), \Psi^*(y)\} = -\mathrm{i}\delta(x-y)$$

apart from the factor $-\mathrm{i}$, the action functional $\mathcal{S}$ for the model (A.1)–(A.3) can be written as

$$\mathcal{S} = \int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}x\, L(x,t)$$

with the Lagrangian density $L(x,t)$ given by

$$L = -\mathrm{i}\Psi^*\dot{\Psi} - H_{\mathrm{LL}}.$$

Next, rewrite this Lagrangian density in terms of another pair of variables $q(x)$ and $q^*(x)$ which gives the stereographic projection of the sphere $\mathbf{S}^2$:

$$S_1 + \mathrm{i}S_2 = \frac{2q}{1+qq^*} \qquad S_1 - \mathrm{i}S_2 = \frac{2q^*}{1+qq^*} \qquad S_3 = \frac{1-qq^*}{1+qq^*}$$

or, equivalently

$$\Psi = \left[\frac{2q}{q^*(1+qq^*)}\right]^{1/2} \qquad \Psi^* = \left[\frac{2q^*}{q(1+qq^*)}\right]^{1/2}.$$

The resulting expression takes the form

$$L \simeq -\frac{2\mathrm{i}\dot{q}}{q(1+qq^*)} - 2\frac{(q'/q)'}{(1+qq^*)} + 2\frac{q^*(q')^2}{q(1+qq^*)^2} - \frac{1}{2}\frac{(J_2-J_1)(q^2+q^{*2}) + 6J_3qq^*}{(1+qq^*)^2}. \tag{A.4}$$

Equivalence $\simeq$ implies, that we ignore terms having the structure of full derivatives in time or in coordinate. It should be pointed out, that variables q and q^* which were complex conjugate to each other when $\boldsymbol{S}(x) \in \mathbb{R}^3$, become independent in the considered complex case $\boldsymbol{S}(x) \in \mathbb{C}^3$. Varying the action corresponding to the Lagrangian density (A.4) with respect to q^*, one obtains

$$(-\mathrm{i}\dot{q} - q'')(1+qq^*) + 2q^*(q')^2 - \frac{J_2-J_1}{2}(q^3 - q^*) + \tfrac{3}{2}J_3q(1-qq^*) = 0. \tag{A.5}$$

This representation of the Landau–Lifshitz equation has been widely used in the study of its soliton and multisoliton solutions (see [18] and references therein). Eliminating $q^*(x)$ from the Lagrangian density (A.4) by the use of (A.5), we find after some algebra

$$L \simeq -\frac{1}{2}\frac{\dot{u}^2 - [u''(x) + \frac{1}{2}(dP(u)/du)]^2}{[u'^2(x) + P(u)]} - \frac{(J_2-J_1)u^2}{2}. \tag{A.6}$$

Here we have denoted

$$P(u) = \frac{J_2-J_1}{4}(1+u^4) - \frac{3J_3}{2}u^2 \qquad u = 1/q.$$

Rescaling of variables

$$u \to \sqrt{k}u \qquad x \to \sqrt{\frac{k}{J_2-J_1}}x \qquad t \to \frac{k}{J_2-J_1}t \qquad L \to -\sqrt{\frac{J_2-J_1}{k}}L$$

with k obeying the relation $k^2 - \frac{6J_3}{J_2-J_1}k + 1 = 0$ tranforms (A.6) into

$$L \simeq \frac{1}{2}\left\{\frac{\dot{u}^2 - [u''(x) + \frac{1}{8}\,\mathrm{d}R(u)/\,\mathrm{d}u]^2}{[u'^2(x) + \frac{1}{4}R(u)]} + k^2u^2\right\}$$

where $R(u)$ is given by (4.21). This is just the Lagrangian density (4.20) obtained in section 4.

Appendix B

Here we obtain explicit expressions for matrices $U(z; x, \tau)$, $V(z; x, \tau)$ defined by (4.5), (4.6).

Let us expand these matrices together with the elliptic function $p(z)$ at the point $z = \omega$:

$$U(z) = \frac{\mathrm{i}\sigma_3}{2(z-\omega)} + \sum_{n=0}^{\infty}(z-\omega)^n U^{(n)} \tag{B.1}$$

$$V(z) = -\frac{\sigma_3}{2(z-\omega)^2} + \sum_{n=-1}^{\infty}(z-\omega)^n V^{(n)} \tag{B.2}$$

$$p(z) = -\frac{1}{z-\omega} + \sum_{n=0}^{\infty}(z-\omega)^{2n+1}\alpha_{2n+1}. \tag{B.3}$$

Substitution of expansions (B.1)–(B.3), (3.6) into (4.5), (4.6) allows one to express the coefficient matrices $U^{(n)}$, $V^{(n)}$ in terms of X_n. The first few equalities are

$$V^{(-1)} = \mathrm{i}U^{(0)} = -\tfrac{1}{2}[\sigma_3, X_1] \tag{B.4}$$

$$V^{(0)} = \Gamma - \frac{\alpha_1\sigma_3}{2} \tag{B.5}$$

$$U^{(1)} = -\mathrm{i}\Gamma - \frac{\mathrm{i}\alpha_1\sigma_3}{2} + \frac{\partial X_1}{\partial x} \tag{B.6}$$

where

$$\Gamma = -\frac{1}{4}\frac{\partial^2}{\partial z^2}\bigg|_{z=\omega}(X^{-1}(z)\sigma_3 X(z)) = \tfrac{1}{4}[X_2, \sigma_3] + \tfrac{1}{2}(X_1\sigma_3 X_1 - X_1^2\sigma_3). \tag{B.7}$$

Relations (4.10), (4.11) give alternative (finite) expansions for elliptic matrices $U(z)$, $V(z)$. Taking into account (4.12) and (4.3), one can determine projective operators P_j in (4.10) up to two parameters z_1 and α:

$$P_1 = r\begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta} & \mathrm{i}\mathrm{e}^{\alpha} \\ -\mathrm{i}\mathrm{e}^{-\alpha} & \mathrm{e}^{-\mathrm{i}\theta} \end{pmatrix} \qquad P_2 = r\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\theta} & \mathrm{i}\mathrm{e}^{\alpha} \\ -\mathrm{i}\mathrm{e}^{-\alpha} & \mathrm{e}^{\mathrm{i}\theta} \end{pmatrix}. \tag{B.8}$$

Here r and θ are related to z_1' by

$$r\cos\theta = \tfrac{1}{2} \qquad r\sin\theta = (4z_1')^{-1}. \tag{B.9}$$

One-dimensional images of operators P_j are generated by vectors e_j

$$e_j = \begin{pmatrix} \gamma_j \\ 1 \end{pmatrix} \qquad P_j e_j = e_j \qquad j = 1, 2 \tag{B.10}$$

where $\gamma_{1,2} = \mathrm{i}\exp(\pm\mathrm{i}\theta + \alpha)$.

Comparing (4.11) with (B.2), one can see, that $B = V^{(-1)}$. It is evident from (B.4), that diagonal entrances of this matrix are zero. Introducing one new parameter l, we can determine matrices L_j and B in (4.11)

$$\begin{gathered} L_1 = \begin{pmatrix} l & (1-l)\gamma_1 \\ l/\gamma_1 & (1-l) \end{pmatrix} \qquad L_2 = \begin{pmatrix} l & (1-l)\gamma_2 \\ l/\gamma_2 & (1-l) \end{pmatrix} \\ B = V^{(-1)} = 2\dot{z}_1\sin\theta\begin{pmatrix} 0 & (l-1)\mathrm{e}^{\alpha} \\ -l\mathrm{e}^{-\alpha} & 0 \end{pmatrix}. \end{gathered} \tag{B.11}$$

Here we have taken into account relations (4.13), (4.14), (B.10).

Matrices U_0 and V_0 can be obtained from (B.4), (B.5):

$$U_0 = -\mathrm{i}B + z_1' P_1\zeta(\omega - z_1) + z_2' P_2\zeta(\omega - z_2)$$

$$V_0 = \Gamma - \frac{\alpha_1\sigma_3}{2} + \dot{z}_1 L_1\zeta(\omega - z_1) + \dot{z}_2 L_2\zeta(\omega - z_2)$$

where the matrix Γ defined by (B.7) takes the form

$$\Gamma = \begin{pmatrix} B_{12}B_{21} & \mathrm{i}B_{12}' - z_1' r\mathrm{e}^{\alpha}[\wp(\omega - z_1) - \wp(\omega - z_2)] \\ -\mathrm{i}B_{21}' + z_1' r\mathrm{e}^{-\alpha}[\wp(\omega - z_1) - \wp(\omega - z_2)] & -B_{12}B_{21} \end{pmatrix}. \tag{B.12}$$

Relations (B.4), (B.6) have been used in deriving (B.12).

Thus, relations (4.10), (4.11), (4.3), (B.8)–(B.12) give explicit expressions for matrices $U(z; x, \tau)$, $V(z; x, \tau)$. These matrices depend on x and τ via three functions $z_1(x, \tau), \alpha(x, \tau)$, and $l(x, \tau)$.

Let us substitute the obtained expressions for the matrices $U(z; x, \tau)$, $V(z; x, \tau)$ into the compatibility condition (4.9). Relations between functions $z_1(x, \tau), \alpha(x, \tau), l(x, \tau)$ can be established then by putting equal to zero singular and constant in z terms in (4.9). One can verify, that all singular terms in (4.9) at the point $z = \omega$ vanish identically. Furthermore, terms proportional to $(z - z_j)^{-2}$, $j = 1, 2$ vanish as well. Setting residues in z_1, z_2 of (4.9) equal to zero, we derive equations (4.15), (4.16), and equality

$$(2l - 1)\dot{z}_1 = 2rz_1' f(z_1) \tag{B.13}$$

where

$$r = \frac{[1 + 4(z_1')^2]^{1/2}}{4z_1'} \qquad f = \frac{z_1''}{2rz_1'} + 2r\frac{\partial \ln \lambda}{\partial x} \qquad \lambda = \wp(z_1 - \omega) - \wp(z_1). \tag{B.14}$$

Relation (B.13) allows us to eliminate the function $l(x, \tau)$ from matrices $U(z; x, \tau)$, $V(z; x, \tau)$. Lastly, the constant term in the Laurent expansion of (4.9) near $z = \omega$ does not give new constrains on $z_1(x, \tau), \alpha(x, \tau)$ vanishing identically.

Appendix C

Here we give explicit expressions for matrices $U(z; x) \equiv U(z; x, \tau)_{\tau=0}$ and $W(z; x)$ associated with the Riemann–Hilbert problem II for $\tau = 0$, $x > 0$.

Projectors P_1 and P_2 in (4.10) are still described by relations (B.8), where parameter α is put equal to zero:

$$\exp \alpha = 1. \tag{C.1}$$

The other coefficient matrices in (4.10), (5.3) are as follows.

$$\begin{gathered}
Q_1 = \rho \begin{pmatrix} e^{i\varphi} & ie^{i(\theta-\varphi)} \\ -ie^{i(-\theta+\varphi)} & e^{-i\varphi} \end{pmatrix} \qquad Q_2 = \rho \begin{pmatrix} e^{-i\varphi} & ie^{i(-\theta+\varphi)} \\ -ie^{i(\theta-\varphi)} & e^{i\varphi} \end{pmatrix} \\
A = I - Q_1 - Q_2 \\
U_0 = -\frac{dz_1}{dx} P_1 \zeta(z_1 - \omega) - \frac{dz_2}{dx} P_2 \zeta(z_2 - \omega) + \frac{Q_1 + Q_2 - I}{x} \\
W_0 = Ia_0 + \sigma_1 a_1 + \sigma_2 a_2 + \sigma_3 a_3
\end{gathered} \tag{C.2}$$

where

$$\begin{aligned}
a_0 &= -\frac{3\eta}{2} \qquad a_1 = \rho \sin(\theta - \varphi)[\zeta(z_2 - \omega) - \zeta(z_1 - \omega)] + \frac{d}{dx}[2\rho\cos(\theta - \varphi)] \\
a_2 &= -\rho\cos(\theta - \varphi)[\zeta(z_1 - \omega) + \zeta(z_2 - \omega)] + rx\frac{dz_1}{dx}\lambda \\
a_3 &= \frac{ix}{4}[\wp(z_1 - \omega) + \wp(z_2 - \omega)] - \frac{i\tan\varphi}{2}[\zeta(z_2 - \omega) - \zeta(z_1 - \omega)] \\
&\quad + \frac{i\rho\cos(\theta - \varphi)}{4r}\left(\frac{dz_1}{dx}\right)^{-2}\frac{d\ln\lambda}{dx} + \frac{i}{r}\frac{d[\rho\cos(\theta - \varphi)]}{dx}.
\end{aligned} \tag{C.3}$$

Here parameters ρ and φ are related by

$$\rho\cos\varphi = \tfrac{1}{2} \qquad \rho\cos(\theta - \varphi) = -\frac{x}{4}f.$$

Parameters r, θ, f, λ are defined by (B.9), (B.14). Operators Q_j, $j = 1, 2$ are projectors: $Q_j^2 = Q_j$. Their images coincide with those of P_j: $\mathrm{Im}\, Q_j = \mathrm{Im}\, P_j = e_j \cdot \mathbb{C}$.

All matrices and parameters listed above depend on x via the single function $z_1(x)$. Compatibility condition (5.2) gives rise to the following nonlinear ordinary differential equation of the fourth order on the function $z_1(x)$:

$$\frac{d}{dx}\left[xf^2 - \frac{xf}{4r(z_1')^2}\frac{d\ln\lambda}{dx} - \frac{f}{r} - \frac{x}{r}\frac{df}{dx}\right] + x\frac{d}{dx}[\wp(z_1) + \wp(z_1 - \omega)] + f^2 = 0. \tag{C.4}$$

This equation can be written in the Lagrangian form (5.4), (5.5).

Representations described above can be derived from (4.5), (4.10), (5.1), (5.2), (5.3) in the same manner, analogous to the results in appendix A. Here we shall prove only statement (C.1), and obtain the boundary condition for equation (C.4) in the origin.

Let $\mathbb{C}^2$ be the two-dimensional linear space of vectors (columns), where matrices $X(z;x)$, $U(z;x)$ operate as linear operators. Equip $\mathbb{C}^2$ by the inner product $\langle\,,\rangle$ defined by

$$\langle c, c'\rangle = c_1 c_1' + c_2 c_2' \qquad c, c' \in \mathbb{C}^2 \qquad c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \qquad c' = \begin{pmatrix} c_1' \\ c_2' \end{pmatrix}.$$

Let us consider the matrix $\Xi(z;x)$ given by

$$\Xi(z;x) = X^T(-z;x)X(z;x)$$

where $X^T(-z;x)$ is the transpose of the matrix $X(-z;x)$. It is easy to derive from properties of $X(z;x)$ stated in problem II, that the matrix $\Xi(z;x)$ is elliptic in z with periods $2\omega, \omega'$. Its single (second-order) pole in Γ_1 lies in the origin, and

$$\Xi(\omega;x) = I \qquad \ker \Xi(z_j;x) = e_j \cdot \mathbb{C} \qquad j = 1, 2.$$

The second equality follows from (4.14), (B.10). Summarizing the above properties of $\Xi(z;x)$, we can rewrite it in the form

$$\Xi(z;x) = I + \Xi_0(x)[\wp(z) - \wp(\omega)]$$

where $\Xi_0(x)$ is the symmetric matrix $\Xi_0^T(x) = \Xi_0(x)$ with eigenvectors e_j and eigenvalues $[\wp(\omega) - \wp(z_j)]^{-1}$, $j = 1, 2$. Since these eigenvalues, generally speaking, do not coincide, eigenvectors e_j are orthogonal to each other, i.e. $0 = \langle e_1, e_2\rangle = 1 - \mathrm{e}^{2\alpha}$. So, only two values for e^{α} are allowed

$$\mathrm{e}^{\alpha} = 1 \qquad \text{or} \qquad \mathrm{e}^{\alpha} = -1 \tag{C.5}$$

and α does not depend on x. The second possibility in (C.5) can be eliminated by evaluation of matrices $X(z;x)$, $U(z;x)$ near the point $x = 0$, which provides, also, the boundary condition for the differential equation (C.4).

At $x = 0$, $\tau = 0$ problem II can be solved explicitly in quasiperiodic ζ-functions (4.2):

$$X(z;0) = \Upsilon(z) \begin{pmatrix} \rho(z) & -\mathrm{i}[\rho(z) + \mathrm{i}\pi\omega^{-1}] \\ -\mathrm{i}[\rho(z) + \mathrm{i}\pi\omega^{-1}] & -[\rho(z) + 2\mathrm{i}\pi\omega^{-1}] \end{pmatrix} \tag{C.6}$$

where

$$\Upsilon(z) = \frac{\mathrm{i}\omega}{\pi} \frac{\vartheta_1[(z - z_0)/2\omega]}{\vartheta_1(z/2\omega)} \frac{\vartheta_1(\frac{1}{2})}{\vartheta_1(\frac{1}{4})}$$

$$\rho(z) = [\zeta(z - z_0) - \zeta(\omega - z_0)] - \frac{(z - \omega)\eta + \pi\mathrm{i}}{\omega}$$

$$z_0 = \omega/2 \qquad \eta = \zeta(\omega).$$

Matrix (C.6) degenerates at $z = z_0 = \omega/2$: $X(z_0;0) \sim \sigma_3 - \mathrm{i}\sigma_1$. So, at $x = 0$ points $z_1(x)$ and $z_2(x)$ merge in z_0. For small x we assume the expansion

$$z_{1,2}(x) = z_0 \pm \varsigma\sqrt{x}(1 + \nu x + \cdots). \tag{C.7}$$

Substituting (C.7) and (B.8) into (4.10) we find for $x = 0$

$$U(z;0) = \frac{\mathrm{i}}{2}\sigma_3[\zeta(z - \omega) - \zeta(z - z_0)] - \frac{\varsigma^2}{2}\wp(z - z_0)(I - \mathrm{e}^{\alpha}\sigma_2) + U_0.$$

Now, let us rewrite relation (4.5) in the form

$$\frac{\partial X}{\partial x} = XU + \frac{\mathrm{i}p}{2}\sigma_3 X \tag{C.8}$$

and set $x = 0$ in it. The left-hand side is analytical in z at the points z_0 and ω. The right-hand side has the second-order pole in z_0, and the first-order pole in ω. Equating to zero corresponding singular terms in the right-hand side of (C.8), we obtain

$$\begin{aligned}
&\frac{1}{(z-z_0)^2} : \ \mathrm{e}^{\alpha} = 1 \qquad \text{at } x = 0 \\
&\frac{1}{(z-z_0)} : \ \varsigma = \sqrt{\omega/\pi} \\
&\frac{1}{(z-\omega)} : \ \nu = \frac{\eta}{2\pi} - \frac{\pi}{8\omega}.
\end{aligned} \tag{C.9}$$

Combining (C.9) with (C.5) we complete proof of (C.1). Expansion (C.7) with obtained ς and ν gives the boundary condition (5.8) for the differential equation (C.4).

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